APPROXIMATE SOLUTION OF THE INTERNAL PROBLEM FOR A TURBULENT BOUNDARY LAYER

A. Sh. Dorfman and O. D. Lipovetskaya

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The development of a turbulent boundary layer in an axisymmetric channel is analyzed with an an account of the mutual effects of the boundary layer and the core of the flow. The coordinates of the separation point, the coordinates of the junction, and other characteristics are found as functions of the Reynolds number and the channel geometry. A simple approximate method is proposed for calculating the boundary layer in a channel with an arbitrary curvilinear generatrix.

In solving internal problems in boundary-layer theory one must take into account the mutual effects of the boundary layer and the core of the flow. These effects are usually taken into account by the method of successive approximations [1, 2]; in several cases the calculation must be repeated several times or carried out for parts of the channel, rather than for the channel as a whole, in order to achieve the necessary accuracy. In particular, this is true of calculations carried out for divergent channels [2].

Below we solve the problem of the development of a turbulent boundary layer in an axisymmetric channel, taking account of the mutual effects, by reducing the problem to a single integro-differential equation, as was done for planar channels and laminar flow in [3]. To calculate the characteristics of the boundary layer we will use the integral method of [4], which is based on a joint solution of the integral momentum and energy relations and the use of experimental data. In this approach, the problem is described by:

$$\frac{d\delta^{**}}{dx}+\frac{\delta^{**}}{U}\frac{dU}{dx}(2+H)+\frac{1}{r}\frac{dr}{dx}\delta^{**}=\frac{\tau_w}{\rho U^2}; \tag{1}$$

$$\frac{1}{U^3 r}\frac{d}{dx}(U^3 r H^* \delta^{**})=2\int_0^{\delta}\frac{\tau}{\rho U^2}\frac{y}{r}\frac{\partial}{\partial y}\left(\frac{u}{U}\right)dy; \tag{2}$$

$$\frac{Q}{2\pi}=Ur^2\left(\frac{1}{2}-\frac{\delta^*}{r}\right), \tag{3}$$

i.e., the integral momentum and energy relations and the divergence equation, respectively. As in [3], we are using a coordinate system whose x axis coincides with the symmetry axis of the channel. As was shown in [3], the equations written in terms of this coordinate system are essentially the same as the ordinary equations, written in terms of the coordinate system in which the x axis is reckoned along the surface in the flow.

In Eqs. (1)-(3), the characteristic boundary-layer thicknesses are defined by

$$\delta^*=\int_0^r\left(1-\frac{u}{U}\right)\frac{y}{r}dy;\quad \delta^{**}=\int_0^r\left(1-\frac{u}{U}\right)\frac{u}{U}\frac{y}{r}dy;\quad \delta^{***}=\int_0^r\left[1-\left(\frac{u}{U}\right)^2\right]\frac{u}{U}\frac{y}{r}dy.$$

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Using these characteristic thicknesses, we can convert the integral relations for axisymmetric flow [2, 6], written with an account of the effect of transverse curvature, to the form of the ordinary equations, written under the assumption that the boundary layer is thin in comparison with the radius curvature of the surface. We can then use the following experimental dependences to close system (1)-(3), as in [2, 4]:

$$\frac{\tau_w}{\rho U^2}=\frac{0.123}{\mathrm{Re}_{**}^{0.268}}\,10^{-0.678H};\quad \int_0^\delta \frac{\tau}{\rho U^2}\,\frac{y}{r}\,\frac{\partial}{\partial y}\left(\frac{u}{U}\right)dy=\frac{0.0056}{\mathrm{Re}_{**}^m};$$
$$H^*=\frac{1.269H}{H-0.379};\quad \mathrm{Re}_{**}=\frac{U\delta^{**}}{\nu}. \tag{4}$$

Following the procedure of [4], and using the second of Eqs. (4), we convert Eq. (2) to

$$S=S_1+A\int_0^\varphi \frac{\bar r^{m+1}}{z^{3+2m}}\,d\varphi, \tag{5}$$

where the subscript 1 refers to the inlet cross section, and where

$$S=\left(\frac{\bar\delta^{**}\bar r^2}{z^3}\right)^{m+1};\quad z=\frac{Q}{\pi r_1^2 U}=\bar r^2(1-2\bar\delta^*);$$
$$\varphi=\frac{x}{r_1\mathrm{Re}_1^m};\quad \mathrm{Re}=\frac{Ur}{\nu};\quad \bar r=\frac{r}{r_1};\quad \bar\delta^*=\frac{\delta^*}{r};\quad \bar\delta^{**}=\frac{\delta^{**}}{r}. \tag{6}$$

The constant A is determined by comparing Eq. (5) with experimental data for a plate; it is equal to A = 0.016 at Re values for which the Blasius equation holds (m = 1/4) or A = 0.0076 at Re values for which the Faulkner equation holds (m = 1/6) [5].

Divergence equation (3) can be written in terms of notation (6) as

$$H=\frac{\bar r^2-z}{2z^3S^{\frac{1}{m+1}}}. \tag{7}$$

To obtain a third equation to close system (5), (7) with the three unknowns (S, z, H), we multiply Eq. (1) by H* and subtract the result from Eq. (2), finding

$$\mathrm{Re}_{**}^m\delta^{**}\frac{dH^*}{dx}=(H-1)H^*\frac{\mathrm{Re}_{**}^m\delta^{**}}{U}\frac{dU}{dx}+f,$$

where

$$f=\left[2\int_0^\delta \frac{\tau}{\rho U^2}\,\frac{y}{r}\,\frac{\partial}{\partial y}\left(\frac{u}{U}\right)dy-H^*\frac{\tau_w}{\rho U^2}\right]\mathrm{Re}_{**}^m. \tag{8}$$

Using Eq. (4) and carrying out some straightforward manipulations, we convert Eq. (8) to

$$\frac{dH}{d\varphi}=F_1\frac{1}{z}\frac{dz}{d\varphi}+F_2\frac{\bar r^{m+1}}{Sz^{2m+3}}, \tag{9}$$

where

$$F_1=\frac{H^*(H-1)}{dH^*/dH}=2.64H(H-1)(H-0.379)$$

is a function of only H. The function

$$F_2=\frac{f}{dH^*/dH}=0.324(H-0.379)\left[\frac{H\cdot 10^{-0.678H}}{\mathrm{Re}_{**}^{0.268-m}}-0.072(H-0.379)\right],$$

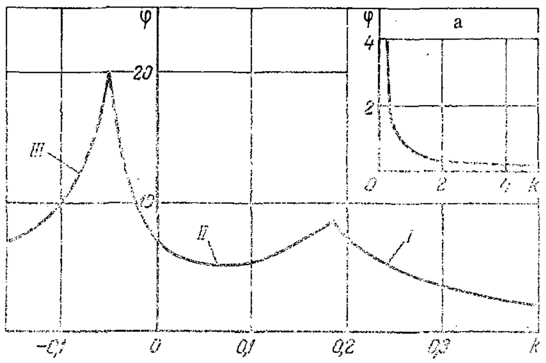

Fig. 1

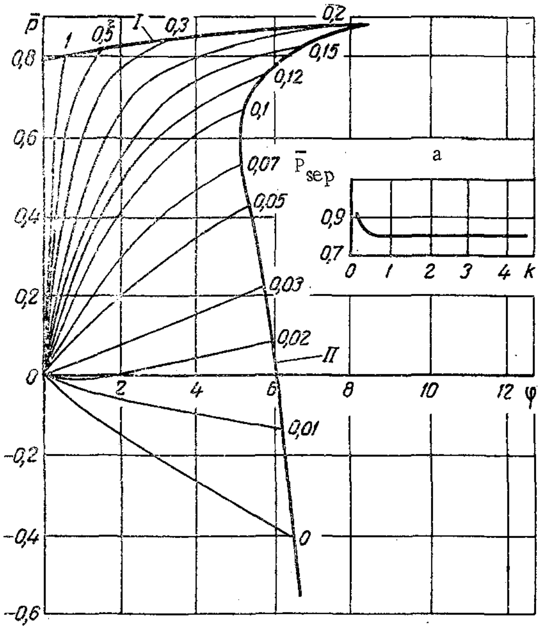

Fig. 2

Fig. 1. Dependence of the dimensionless coordinates $\varphi = x/r_1 Re_1^{1/6}$ of the separation point and junction on the number $k = Re_1^{1/6} \tan\theta/2$: I) separation line; II) junction line; III) no-junction line; a) dependence of the dimensionless coordinate of separation point on k for large k.

Fig. 2. Dependence of the dimensionless static pressure $\overline{P}$ on φ: I) separation line; II) junction line. The numbers show constant k lines; a) dependence of the dimensionless pressure $\overline{P}_{sep}$ at the separation point on k.

generally depends on only H and Re_{**}. However, because of the small value of the exponent $(0.268 - m)$, which is equal to 0.018 at $m = 1/4$ and 0.1 at $m = 1/6$, the Re_{**} dependence of F_2 is weak and can be neglected; i.e., we can calculate F_2 for some average Re_{**}. For the subsequent calculations we use the F_2 values corresponding to $Re_{**} = 10^4$. Following the procedure of [3], we can reduce system (5), (7), (9) to a single integro-differential equation for the function z.

For a computer calculation, however, it is more convenient to write system (5), (7), (9) as two differential equations; the first is obtained by differentiating Eq. (5), while the second is obtained by differentiating Eq. (7), eliminating $dH/d\varphi$ by means of Eq. (9), and then solving the resulting equation for $dz/d\varphi$. The result is

$$S' = A\,\frac{\overline{r}^{m+1}}{z^{3+2m}};$$

$$z' = \frac{z\left[2(m+1)\,\overline{r}\,\overline{r}'z^{3+2m}S - A\,\overline{r}^{m+1}(\overline{r}^2 - z) - 2(m+1)\,F_2\,\overline{r}^{m+1}z^3 S^{\frac{1}{m+1}}\right]}{(m+1)\,z^{3+2m}S\left[2F_1 z^3 S^{\frac{1}{m+1}} + 3(\overline{r}^2 - z) + z\right]}, \tag{10}$$

where the prime denotes differentiation with respect to φ. This system must be integrated with the initial conditions

$$\varphi = 0,\ z_1 = 1 - 2\overline{\delta}_1^*,\ \ S_1 = \left[\frac{\overline{\delta}_1^{**}}{(1-2\overline{\delta}_1^*)^3}\right]^{m+1}.$$

In the case in which the generatrices of the channel are straight and form an angle θ we have

$$\overline{r} = 1 + k\varphi;\quad k = Re_1^m\ \mathrm{tg}\,\frac{\theta}{2},$$

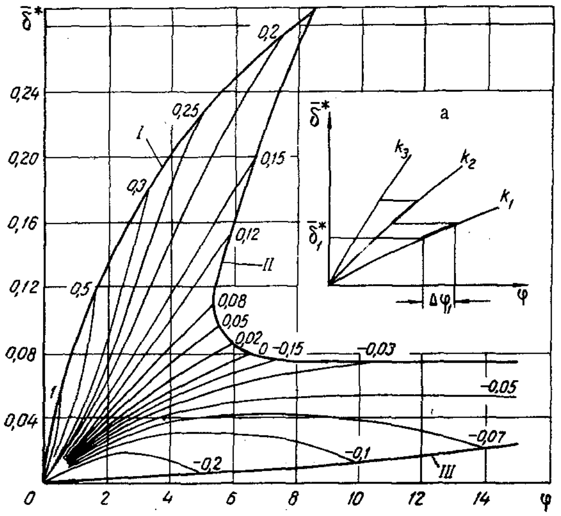

Fig. 3

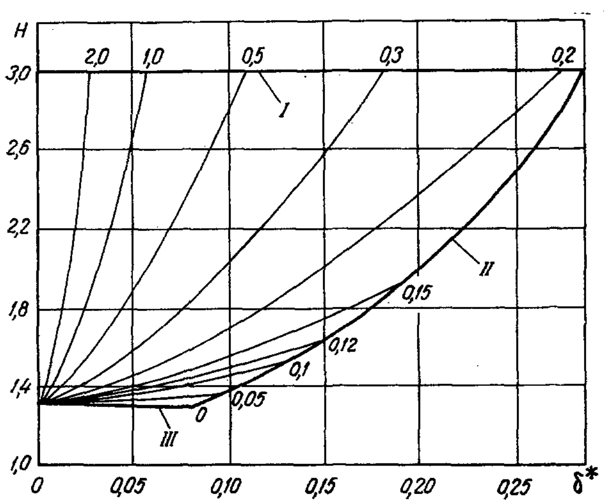

Fig. 4

Fig. 3. Dependence of $\bar{\delta}^*$ on φ: I) separation line; II) junction line; III) no-junction line. The numbers show constant k lines; a) diagram used to design a channel with a curvilinear generatrix.

Fig. 4. Diagram used to determine H: I) separation line; II) junction line; III) no-junction line. The numbers show constant k lines.

where $k > 0$ corresponds to divergent channels and $k < 0$ corresponds to convergent channels. If the velocity profile at the entrance to the channel is uniform ($\bar{\delta}_1^* = \bar{\delta}_1^{**} = 0$), all the quantities involved turn out to depend on only k and can be conveniently tabulated. We integrated system (10) for this case with $m = 1/6$ on a computer by the Runge–Kutta method. Since we have $\bar{\delta}_1^* = \bar{\delta}_1^{**} = 0$ in this case, we have $z_1 = 1$ and $S_1 = 0$ in the entrance cross section, and z' cannot be determined from the second equation. The numerical calculation is therefore possible only beginning at some $\varphi > 0$. Near $\varphi = 0$ the solution can be written in series form:

$$z = 1 + a_1\varphi^{\frac{1}{1+m}} + a_2\varphi + a_3^{\frac{1+2m}{1+m}} + \ldots + a_i\varphi^{\frac{1+(i-1)m}{1+m}} + \ldots$$

Substituting this series into system (5), (7), (9), we can determine the coefficients a_i. In particular, for $m = 1/6$ this series becomes

$$z = 1 - 0.0405\varphi^{6/7} + 2k\varphi + 0.00312\varphi^{12/7} - 0.17k\varphi^{13/7} + \ldots$$

The calculation is carried out either up to the separation point or up to the junction point for the boundary layers. The separation point is governed by the separation value of the form parameter [4], H = 3, while the junction is governed by the dimensionless layer thickness, $\delta/r = 1$. To find the latter we use

$$\frac{\delta^*}{r} = \frac{n}{n+1}\,\frac{\delta}{r} - \frac{n}{2(n+2)}\left(\frac{\delta}{r}\right)^2;$$

$$\frac{\delta^{**}}{r} = \frac{n}{(n+1)(2n+1)}\,\frac{\delta}{r} - \frac{n}{2(n+1)(n+2)}\left(\frac{\delta}{r}\right)^2,$$

which corresponds to a power-law velocity distribution in the boundary layer. Substituting $\delta/r = 1$ into these dependences, and eliminating n from the results, we easily find a relation between δ^*/r and H at the junction point. This dependence is shown graphically in Fig. 4 (curve II).

Figure 1 shows the calculated results; the layers are seen to join at $k < 0.18$. The calculation shows that with $k < -0.05$ the layers join near the point at which the walls of the convergent channel meet, so that in practice we can assume that the layers do not join in these channels. With $k > 0.18$ the separation occurs before the layers join. All three regions are shown in Fig. 1; the curve in the region $k < -0.05$ gives the k dependence of the coordinate of the point at which the walls of the convergent channel meet. The dependence of this coordinate on k is interesting, displaying minimum at $k = 0.08$. As k changes in either direction from this value, the coordinate of the junction increases. Since this coordinate increases with increasing k, the rate at which the boundary layer increases along the length of the channel, which is a function of k, increases more slowly than the rate at which the cross-sectional area changes. Analogously, the increase in the coordinate of the junction with decreasing k can be attributed to the relation between the rate at which the boundary layer grows and the rate of change of the cross-sectional area.

Figures 2-4 show the changes along the channels in the dimensionless static pressure $\overline{P} = (P - P_1) / \rho U_1^2/2$, the dimensionless displacement thickness $\overline{\delta}^* = \delta^*/r$, and the form parameter $H = \delta^*/\delta^{**}$. Using these results we can easily determine all the pertinent characteristics of the flow of any cross section. In particular, the drag coefficient in the channel can be determined from [1]

$$\xi = \frac{2\overline{\delta}^{***}}{\overline{r}^2(1-2\overline{\delta}^*)^3}; \quad \overline{\delta}^{***} = \frac{\overline{\delta}^* H^*}{H},$$

where H^* is given by the third of Eqs. (4).

The reduction coefficient for the static pressure is numerically equal to the relative static pressure at the end of the channel, while the total drag coefficient (including the drag at the exit velocity) is

$$\xi_n = 1 - \eta.$$

Figure 2 shows the dependence of the dimensionless static pressure $\overline{P}_{sep}$ at the separation point on k (Fig. 2a). We see from this figure that in this case the dependences of $\overline{P}_{sep}$ on Re and θ are slightly stronger than in the case of laminar flow in a planar channel, in which $\overline{P}_{sep}$ is essentially independent of Re and θ and equal to ~0.3 [3]. For turbulent flow $\overline{P}_{sep}$ turns out to be essentially independent only for $k > 1$, for which we have $\overline{P}_{sep} = 0.8$. As k decreases from 1 to 0.18, $\overline{P}_{sep}$ increases to 0.9. For channels with $k < 0.18$, the boundary layers join. Thus a much greater part of the dynamic flow pressure applied to the convergent channel can be reduced during turbulent flow before separation than during laminar flow. This circumstance results from the much more rapid exchange of momentum between the core of the flow and the boundary layer in turbulent flow.

Figure 4 shows the form parameter H along the channel as a function of $\overline{\delta}^*$. The curves for negative values essentially coincide with that for $k = 0$. It follows that for negative k the form parameter H changes slowly along the channel and is essentially equal to the value 1.33 corresponding to gradient-free flow. At large positive k, which corresponds to divergent flow, H increases rapidly along the channel, so that the assumption of constant H, which is frequently used to simplify calculations, turns out to be valid only for convergent flow. Figure 4 also shows the dependence of H on $\overline{\delta}^*$, which corresponds to junction of the boundary layers (curve II). The intersections of this curve with k = const lines yield the values of H and $\overline{\delta}^*$ at the junctions.

Finally, we note that these curves can be used to work out a simple approximate method for calculating the boundary layer in a channel having an arbitrary curvilinear generatrix, similar to the method proposed in [3]. For this purpose the channel must be broken up into regions in each of which θ (and thus k) can be assumed constant. If we now use the standard assumption on which the one-parameter methods are based – that the boundary layer within each of these regions develops as it would in a channel with straight generatrices for the same value of k and for the same value of one of the dimensionless characteristic thicknesses (e.g., $\overline{\delta}^*$) at the entrance, we can use Figs. 3 and 4 to approximately determine all the necessary quantities in any cross section of the channel having a curvilinear generatrix. If the profile at the entrance to the channel is uniform ($\overline{\delta}_1^* = \overline{\delta}_1^{**} = 0$), we can use the known value of k to find the corresponding point, 1, on the curve (Fig. 3). Moving a distance $\Delta\varphi_1$ (the dimensionless length of the first region) along the k = const curve beyond point 1, we find the value of $\overline{\delta}_2^*$ in the channel of the corresponding region. Now moving along a horizontal line to the curve k = const which corresponds to k_2, we find point 2, which corresponds to the beginning of the second region, etc. The form parameter in any cross section is determined with the help of Fig. 4 from the known $\overline{\delta}^*$. Knowing $\overline{\delta}^*$ and H, we can easily determine the separation point, the junction, and all other characteristics. If the profile at the entrance to the channel is nonuniform ($\overline{\delta}_1^* \neq 0$; $\overline{\delta}_1^{**} \neq 0$), the calculation can be carried out in an analogous manner, but adequate results can be obtained only when the relation among the quantities $\overline{\delta}_1^*$, $H = \overline{\delta}_1^*/\overline{\delta}_1^{**}$, and $k_1 = \mathrm{Re}_1^m \tan\theta/2$ specified at the entrance cross section corresponds to Fig. 4, i.e., only when the value found for any one of these from two specified quantities and Fig. 4 differs little from the specified value for this (third) quantity.

NOTATION

x, y are the coordinates (the x axis coincides with the symmetry axis of the channel);
θ is the angle between the channel walls;
r_1 is the radius of the entrance cross section of the channel;

r	is the radius of the channel cross section;
U	is the flow velocity in the core;
u	is the flow velocity in the boundary layer;
δ	is the boundary-layer thickness;
$\delta^* = \int_0^r \left(1 - \frac{u}{U}\right) \frac{y}{r} dy$	is the displacement thickness;
$\delta^{**} = \int_0^r \left(1 - \frac{u}{U}\right) \frac{u}{U} \frac{y}{r} dy$	is the momentum-loss thickness;
$\delta^{***} = \int_0^r \left[1 - \left(\frac{u}{U}\right)^2\right] \frac{u}{U} \frac{y}{r} dy$	is the energy-loss thickness;
$H = \delta^*/\delta^{**}$; $H^* = \delta^{***}/\delta^{**}$	are the form parameters;
τ	is the shear stress;
τ_w	is the shear stress at wall;
ρ	is the density;
ν	is the kinematic viscosity;
$Re = Ur/\nu$; $Re_{**} = U\delta^{**}/\nu$	are the Reynolds numbers;
m	is the exponent in the tangential-stress expression, equal to 1/4 according to Blasius and 1/6 according to Faulkner;
$\overline{r} = r/r_1$	is the dimensionless radius;
$\overline{\delta}^* = \delta^*/r$; $\overline{\delta}^{**} = \delta^{**}/r$; $\overline{\delta}^{***} = \delta^{***}/r$	are the dimensionless displacement thickness, momentum-loss thickness, and energy-loss thickness;
$\varphi = x/r_1 Re_1^m$	is the dimensionless coordinate;
$z = Q/\pi r_1^2 U = \overline{r}^2(1 - 2\delta^*)$;	
$S = (\overline{\delta}^{**}\overline{r}^2/z^3)^{m+1}$;	
n	is the exponent in the power-law velocity distribution;
Q	is the liquid flow rate.

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